University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2013
Mathematics 1201
Friday 11 January 2013 3:40 – 5:40

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

1. (i) Negate the following formula, and replace it by an equivalent one which does not involve $\neg$, $\wedge$ or $\vee$;

$$(\forall y)(\exists x)\neg\,(P(x)\wedge\neg Q(y)) \bigwedge (\forall x)(\exists y)\,(Q(y)\wedge\neg R(x))\,.$$

(ii) Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that

(a) f is injective ; (b) f is invertible.

Show that an invertible mapping is injective.

Decide, with explanation in each case, whether the following mappings f, g are

a) injective ; b) surjective;

$$f : \mathbb{Z}\to\mathbb{Z}\,,\ \ f(x) = 2x^3 - x \qquad : \qquad g : \mathbb{R}\to\mathbb{R},\ \ g(x) = 2x^3 - x.$$

(iii) Let σ be a permutation of the set $\{1,\dots,n\}$; explain what is meant by saying that (a) σ is a transposition; (b) σ is an adjacent transposition.

Show that any transposition can be written as a product of adjacent transpositions.

(iv) Decompose the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 16 & 11 & 7 & 2 & 12 & 5 & 3 & 15 & 14 & 6 & 8 & 10 & 9 & 13 & 4 & 1 \end{pmatrix}$$

into a product of disjoint cycles and hence compute $\mathrm{sign}(\sigma)$ and $\mathrm{ord}(\sigma)$.

PLEASE TURN OVER

2. Let $\epsilon(r,s)$ be the basic $m \times m$ matrix given by $\epsilon(r,s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. Explain without proof how to calculate the product $\epsilon(r,s)\epsilon(u,t)$.

Define the elementary $m \times m$ matrices

(i) $E(r,s;\lambda)$ $(r \neq s)$; (ii) $\Delta(r,\lambda)$ $(\lambda \neq 0)$; (iii) $P(r,s)$ $(r \neq s)$

in terms of the basic matrices $\epsilon(r,s)$.

For the matrix A below, find A^{-1} and express A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

3. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

In each case below, decide with justification whether the given vectors are linearly independent over $\mathbb{Q}$. If they are not, give an explicit dependence relation between them.

(a) $$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix};$$

(b) $$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

Suppose that the subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V where $n \geqslant 4$ and that $\mathbf{v}_n$, $\mathbf{v}_{n-1}$ can be expressed as linear combinations

$$\mathbf{v}_n = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_{n-1}$$

$$\mathbf{v}_{n-1} = \mathbf{v}_1 + 2\mathbf{v}_2.$$

Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-2}\}$ also spans V.

State the Exchange Lemma, and explain how it is used in formulating the idea of the *dimension* of a vector space.

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4. Let V, W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a linear mapping. Define (a) the kernel, $\mathrm{Ker}(T)$ and

(b) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbb{Q}^6 \to \mathbb{Q}^3$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 3 & -1 & 3 & 1 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

5. Let $T : U \to V$ be a linear map between vector spaces U , V, and let $\mathcal{E} = (e_i)_{1\leqslant i\leqslant m}$ be a basis for U and $\Phi = (\varphi_j)_{1\leqslant j\leqslant n}$ be a basis for V. Explain what is meant by the matrix $\mathcal{M}(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right) and prove that if $S : V \to W$ is also a linear map and $\Psi = (\psi_k)_{1\leqslant k\leqslant p}$ is a basis for W then

$$\mathcal{M}(S \circ T)^{\Psi}_{\mathcal{E}} = \mathcal{M}(S)^{\Psi}_{\Phi}\mathcal{M}(T)^{\Phi}_{\mathcal{E}}.$$

$$\text{Let } \mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \; ; \quad \Phi = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right\}$$

be bases for $\mathbb{F}^3$ and let $T : \mathbb{F}^3 \to \mathbb{F}^3$ be the mapping

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 & -x_2 & \\ & 3x_2 & \\ -2x_1 & +4x_2 & +x_3 \end{pmatrix}.$$

Write down (i) $\mathcal{M}(T)^{\mathcal{E}}_{\mathcal{E}}$ and (ii) $\mathcal{M}(\mathrm{Id})^{\mathcal{E}}_{\Phi}$. Hence find $\mathcal{M}(T)^{\Phi}_{\Phi}$.

PLEASE TURN OVER

6. Let V be the vector space consisting of all functions $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) \quad = \quad \lambda_1 \sin(x) + \lambda_2 \cos(x) + \lambda_3\, x\sin(x) + \lambda_4\, x\cos(x)$$

where $\lambda_i \in \mathbb{Q}$ and let $D : V \to V$ be the linear map $D(f) = \frac{df}{dx}$. Taking

$$\{\sin(x),\ \cos(x),\ x\sin(x),\ x\cos(x)\}$$

as basis for V find :

i) the matrix of D ; ii) the matrix of D^4 ; iii) the matrix of D^{-1}.

Hence *without further explicit differentiation or integration* write down

iv) $\dfrac{d^4}{dx^4}(2\sin(x) - \cos(x) + x\cos(x))$

v) $\displaystyle\int \{\sin(x) + 2x\sin(x) + 3x\cos(x)\}\, dx$

[You may ignore the constant of integration in v)].

END OF PAPER